SYDNEY
BOYS
HIGH
SCHOOL

2018

YEAR 12
THSC
ASSESSMENT
TASK #4

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided with this paper
- Leave your answers in the simplest exact form, unless otherwise stated
- Marks may **NOT** be awarded for messy or badly arranged work
- In Questions 11–14, show ALL relevant mathematical reasoning and/or calculations

Total Marks: 70

Section I – 10 marks (pages 3 - 7)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 8 - 16)

- Attempt Questions 11–14
- Allow about 1 hours and 45 minutes for this section

Examiner:
J. Chan

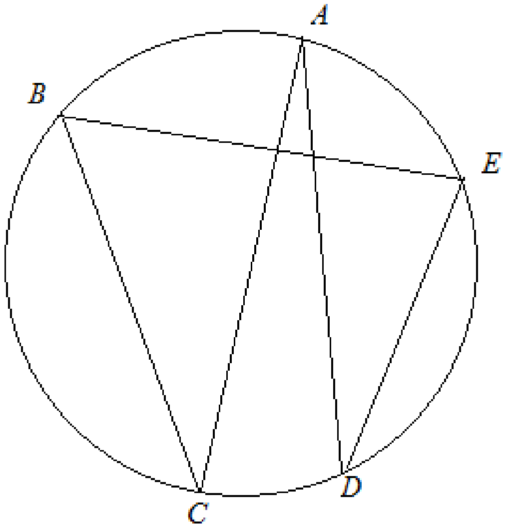
Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

---

1 In the diagram, *AC* is a diameter of the circle *ABCDE*. If $\angle ADE = 28°$, the size of the angle *CBE* is:



A. 56°

B. 62°

C. 72°

D. 76°

2 A particle is moving in simple harmonic motion with displacement x.

Its velocity v is given by $v^2 = 4(25 - x^2)$.

What is the amplitude A of the motion and the maximum speed of the particle?

A. $A = 2$ and maximum speed $v = 5$

B. $A = 2$ and maximum speed $v = 10$

C. $A = 5$ and maximum speed $v = 10$

D. $A = 5$ and maximum speed $v = 5$

3 What is a general solution of $\tan 3x = \tan\alpha$?

A. $x = n\pi + \alpha$, for $n \in \mathbb{Z}$

B. $x = n\pi + \frac{\alpha}{3}$, for $n \in \mathbb{Z}$

C. $x = \frac{n\pi - \alpha}{3}$, for $n \in \mathbb{Z}$

D. $x = \frac{n\pi + \alpha}{3}$, for $n \in \mathbb{Z}$

4 The equation $\sin x = x^2 - 10$ has a root close to $x = \pi$. Use one application of Newton's method to give a better approximation, correct to 4 decimal places.

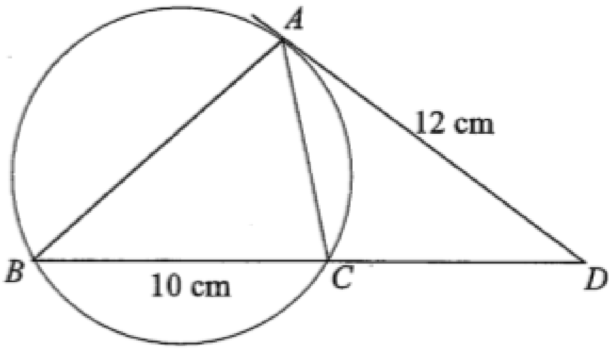
A. –3.1595

B. 3.1595

C. 3.1237

D. –3.1237

5 *ABC* is a triangle inscribed in a circle. The tangent to the circle at *A* meets *BC* produced at *D* where *BC* = 10cm and *AD* = 12cm. What is the length of *CD*?



A. 6 cm

B. 7 cm

C. 8 cm

D. 9 cm

6 Which of the following is the range of the function $y = 2\sin^{-1} x + \frac{\pi}{2}$?

A. $-\pi \le y \le \pi$

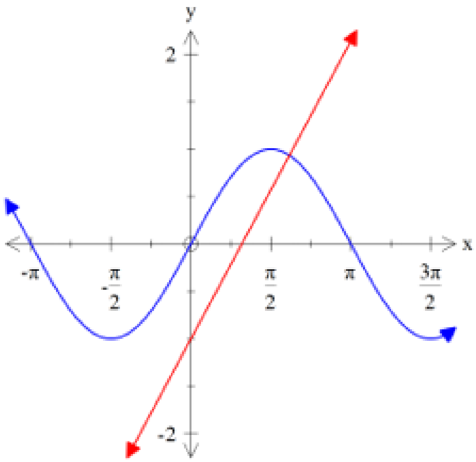
B. $-\pi \le y \le \frac{3\pi}{2}$

C. $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

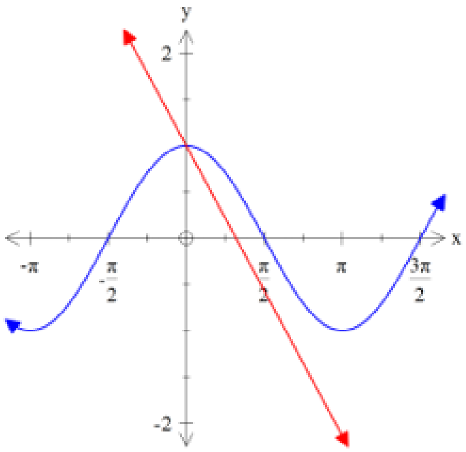
D. $-\frac{\pi}{2} \le y \le \frac{3\pi}{2}$

7 Which of the following graphs could be used to solve $\cos x + x = 0$?

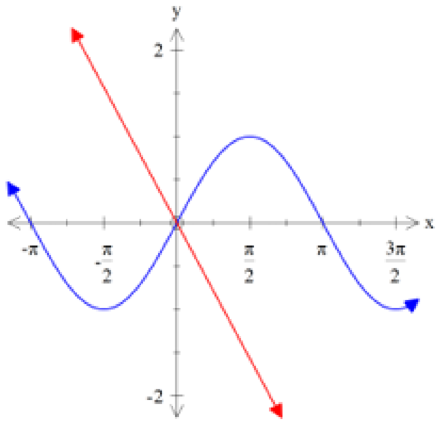
A.



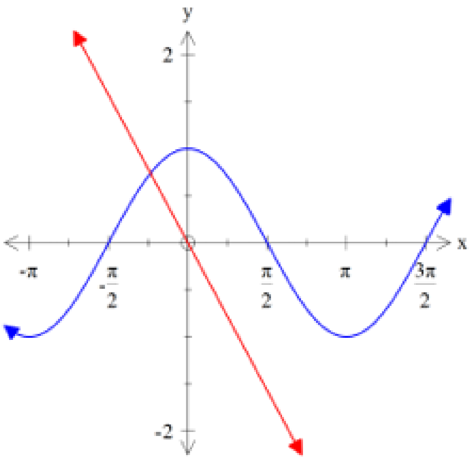
B.



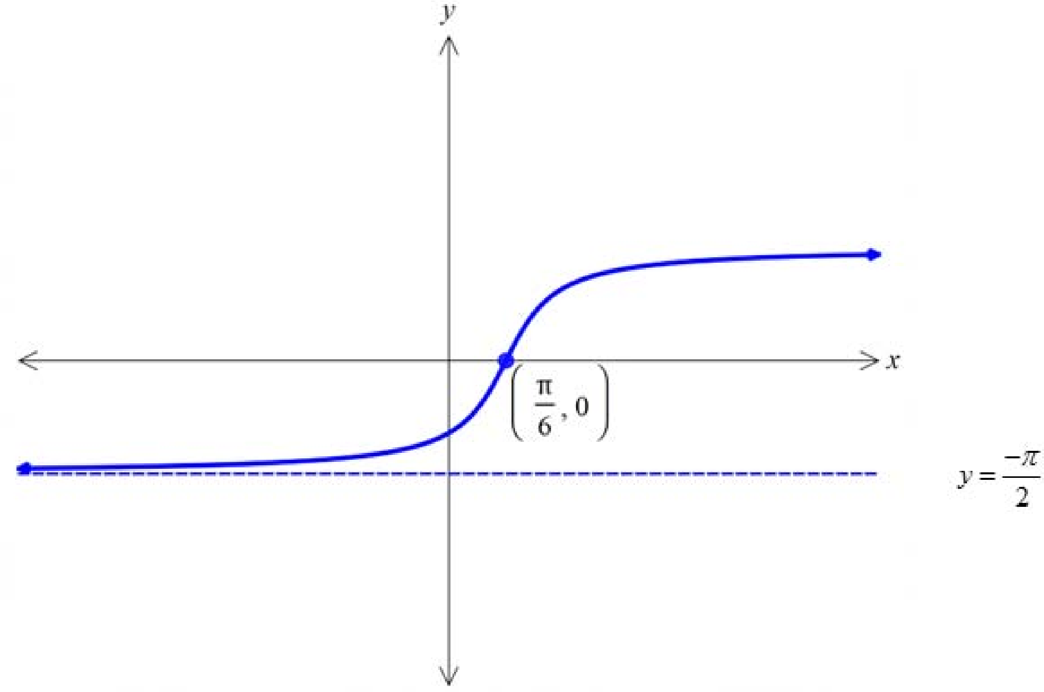
C.



D.



8 Let k be a positive constant and $-\pi \le \theta \le \pi$. If the diagram shows the graph of $y = \tan^{-1}(kx+\theta)$, then:



A. $k = -3$ and $\theta = \frac{\pi}{2}$

B. $k = \frac{1}{3}$ and $\theta = \frac{\pi}{2}$

C. $k = 3$ and $\theta = -\frac{\pi}{2}$

D. $k = \frac{-1}{3}$ and $\theta = -\frac{\pi}{2}$

9 Find $\sin\left(2\tan^{-1}\frac{1}{2}\right)$

A. $\frac{5}{4}$

B. $\frac{3}{5}$

C. $\frac{4}{5}$

D. $\frac{5}{3}$

10 Finding $\lim\limits_{x\to 0}\frac{\sin 4x}{x}$ gives

A. $\frac{1}{4}$

B. 4

C. 2π

D. $\frac{\pi}{8}$

Section II

60 marks

Attempt Questions 11–14

Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Find the derivative of

i) $\dfrac{x}{1+x^2}$ **2**

ii) $\sin(\cos^2 x+e^x)$ **2**

(b) Find the exact value of $\displaystyle\int_0^1\left(e^{-x}+\frac{1}{1+x}+\frac{1}{\sqrt{(1-x^2)}}\right)dx$ **2**

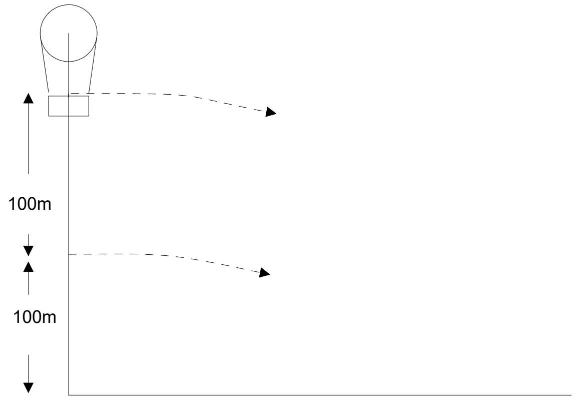
(c) Using the substitution $u=x^4$ or otherwise, show that $\displaystyle\int_0^1\frac{x^3}{1+x^8}dx=\frac{\pi}{16}$ **3**

Question 11 continues on page 9

Question 11 (continued)

(d) Solve $\dfrac{x-3}{x^2+x-2} \geq 0$ **2**

(e) A balloon rises vertically from level ground. Two projectiles are fired horizontally in the same direction from the balloon at a velocity of $80ms^{-1}$. The first is fired at a point 100 m from the ground and the second when it has risen a further 100 m from the ground. Assume the balloon is stationary when the projectiles are launched, how far apart will the projectiles hit the ground? (Use $g = 10ms^{-2}$) **4**



End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) Given that $\int_0^{\frac{\pi}{4}} f(x)dx = 5$ and $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} f(x)dx = 2$, find the value of a given that **2**

$$\int_{\frac{\pi}{2}}^{0} \left(f(x) + a\sin 2x \right) dx = 10$$

(b) The independent term in the expansion of $(2+x)^n$ and the independent term in the expansion of $(2-ax)^{2n+1}$ are in the ratio of 1: 8.

i) Find the value of n. **2**

ii) Hence, given that the coefficient of x^2 in the expansion $(1+x)(2-ax)^{2n+1}$ is 160, find the value(s) of a. **3**

(c) i) Show that $\dfrac{\sin^3\theta + \cos^3\theta}{\sin\theta + \cos\theta} = \dfrac{2-\sin 2\theta}{2}$ **2**

ii) Hence or otherwise, given that $\dfrac{\sin^3\theta + \cos^3\theta}{\sin\theta + \cos\theta} = \dfrac{7}{10}$, show that $\sin 2\theta = \dfrac{3}{5}$ **1**

iii) Given further that 2θ is an acute angle, find the value of tan 3θ. **2**

(d) Use the t-formulae to solve $3\sin\theta - 4\cos\theta = 4$ for $0 \le \theta \le 2\pi$ **3**

Where appropriate, leave your answer correct to 2 decimal places.

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) Given that the coefficient of the $\frac{1}{x^2}$ term in the expansion $\left(\frac{2}{x}-x\right)^6-\left(1+\frac{2}{x}\right)^n$ is 128, **2**

find the value of n.

(b) P is the point $(2at, at^2)$ on the parabola $4ay = x^2$ and ℓ is the tangent at P.

i) Prove that the equation of ℓ is $y = tx - at^2$ **1**

ii) If ℓ cuts the x-axis at A and the y-axis at B, find the coordinates of A and B. **2**

iii) In what ratio does P divide AB? **2**

iv) What is the slope of the line joining P to the focus S? **1**

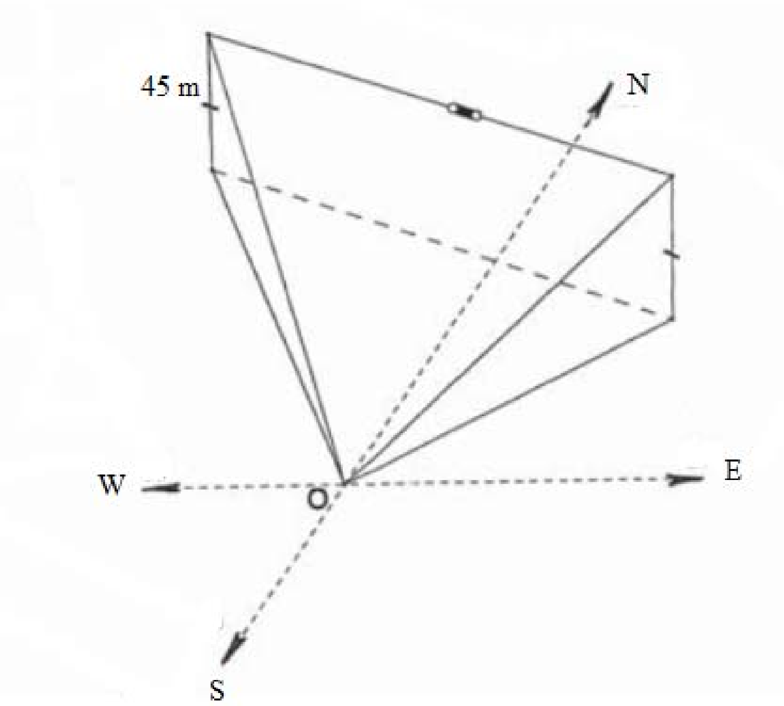
v) Show that ℓ makes equal angles with y-axis and with PS. **2**

(c) A machine produces bolts to meet certain specifications and 90% of the bolts produced meet these specifications. For a sample of 10 bolts, find the probability (in fractions) that exactly 3 fail to meet the specifications. **2**

Question 13 continues on page 13

(d) A cable car is travelling at a constant height of 45m above the ground. An observer on the ground at point *O* sees the cable car on the bearing of 335°*T* from *O* with an angle of elevation of 28°.

After 1 minute the cable car has a bearing of 025°*T* from *O* and a new angle elevation is 53°



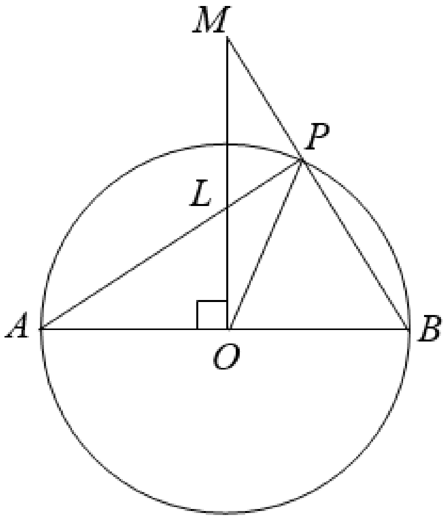
Find the distance, to the nearest metre, the cable car has travelled in that minute **3**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) O is the centre of the circle ABP. $MO \perp AB$. M, P and B are collinear.

MO intersects AP at L.



i) Prove that A, O, P and M are concyclic. **1**

ii) Prove that $\angle OPA = \angle OMB$. **2**

(b) Prove by Mathematical Induction that the sum to n terms of the series

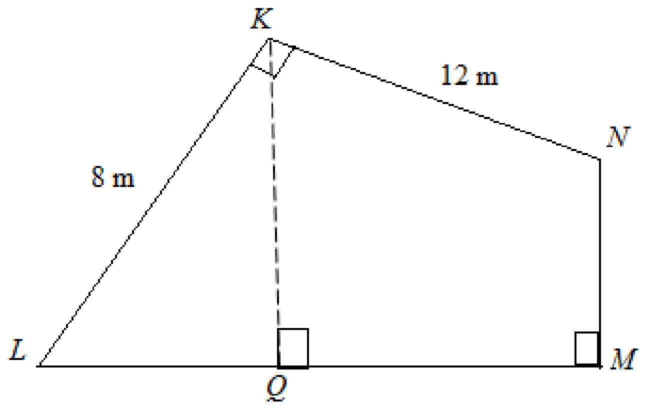
$\log\left(\frac{2}{1}\right)+2\log\left(\frac{3}{2}\right)+3\log\left(\frac{4}{3}\right)+\ldots$ is $\log\frac{(n+1)^n}{n!}$ for n is a positive integer. **3**

Question 14 continues page 15

(c) The diagram shows a pond $KLMN$ in a park. $LK = 8$ metres and $KN = 12$ metres.

$\angle LKN = \angle LMN = 90°$ and $\angle KLM = \theta$, where $0° < \theta < 90°$.

The perimeter of the pond is P metres and $QK \perp LM$.



i) Find the values of the integers a, b and d for which $P = a + b\cos\theta + d\sin\theta$ **2**

ii) Express P in the form of $a + R\sin(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$ **2**

iii) Hence, find the value of θ (to 1 decimal place) when $P = 38$ metres. **1**

Question 14 continues page 16

Question 14 (continued)

(d) A frustum of height H is made by cutting off a right circular cone of base radius r from a right circular cone of base radius R (Figure 1).

It is given that the volume of the frustum is $\frac{\pi}{3}H(r^2 + rR + R^2)$.

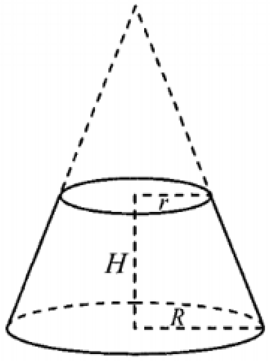


Figure 1

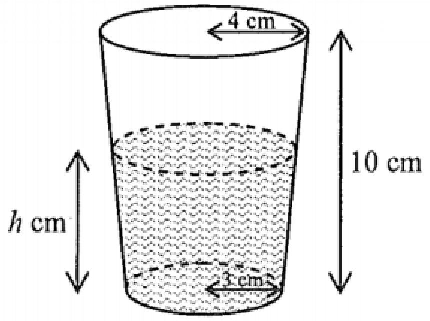


Figure 2

An empty glass is in the form of an inverted frustum described above with the height 10 cm, the radii of the rim and the base 4 cm and 3 cm respectively. Water is being poured into the glass.

Let h cm ($0 \le h \le 10$) be the depth of the water inside the glass at time t second (Figure 2).

i) Show that the volume V cm^3 of water inside the glass at time t is given by

$$V = \frac{\pi}{300}\left(h^3 + 90h^2 + 2700h\right)$$ **2**

ii) If the volume of water in the glass is increasing at the rate 7π cm^3 s^{-1}, find the rate of increase of depth of water at the instant when h = 5cm. **2**

End of paper

SYDNEY
BOYS
HIGH
SCHOOL

2018

YEAR 12
THSC
ASSESSMENT
TASK #4

Mathematics Extension 1

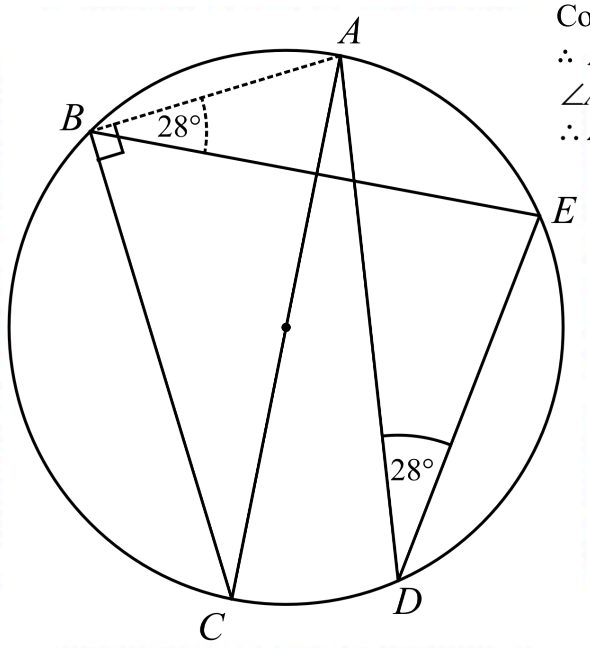
SUGGESTED SOLUTIONS

MC QUICK ANSWERS

1	B
2	C
3	D
4	B
5	C
6	D
7	D
8	C
9	C
10	B

SECTION I MULTIPLE CHOICE SOLUTIONS

1 In the diagram, AC is a diameter of the circle $ABCDE$. If $\angle ADE = 28°$, the size of the angle CBE is:



Construct AB.

$\therefore\ \angle ABE = 28°$ (∠s in the same segment)

$\angle ABC = 90°$ (∠ in a semi-circle)

$\therefore \angle CBE = 90° - 28°$

$= 62°$

A. 56°

[✓ B.] 62°

C. 72°

D. 76°

A	5
B	150
C	6
D	0

2 A particle is moving in simple harmonic motion with displacement x.

Its velocity v is given by $v^2 = 4(25 - x^2)$.

What is the amplitude A of the motion and the maximum speed of the particle?

A. $A = 2$ and maximum speed $v = 5$

B. $A = 2$ and maximum speed $v = 10$

[✓ C.] $A = 5$ and maximum speed $v = 10$

D. $A = 5$ and maximum speed $v = 5$

A	6
B	13
C	140
D	2

SHM about $x = 0$: Amplitude when $v = 0$ and Max speed at $x = 0$

$v = 0 \Rightarrow x^2 = 25$

$\therefore x = \pm 5$

3 What is the general solution of $\tan 3x = \tan \alpha$?

A. $x = n\pi + \alpha$, for $n \in \mathbb{Z}$

B. $x = n\pi + \dfrac{\alpha}{3}$ for $n \in \mathbb{Z}$

C. $x = \dfrac{n\pi - \alpha}{3}$, for $n \in \mathbb{Z}$

[✓ D.] $x = \dfrac{n\pi + \alpha}{3}$, for $n \in \mathbb{Z}$

$\tan \alpha = c \Rightarrow \alpha = n\pi + \tan^{-1} c$

$\therefore \tan 3x = \tan \alpha \Rightarrow 3x = n\pi + \alpha$

$\therefore x = \dfrac{n\pi + \alpha}{3}$

A	**1**
B	**12**
C	**0**
D	**148**

4 The equation $\sin x = x^2 - 10$ has a root close to $x = \pi$. Use one application of Newton's method to give a better approximation, correct to 4 decimal places.

A. –3.1595

[✓ B.] 3.1595

C. 3.1237

D. –3.1237

Let $f(x) = \sin x - x^2 + 10$

$\therefore f'(x) = \cos x - 2x$

$x_1 = \pi$

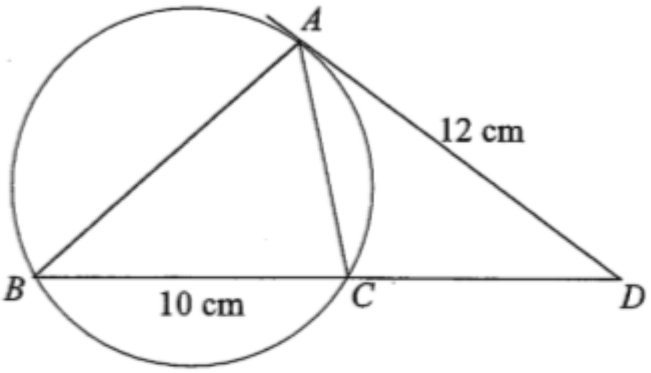
$$x_2 = \pi - \frac{f(\pi)}{f'(\pi)}$$

$$\doteqdot 3.1595$$

A	**0**
B	**158**
C	**1**
D	**1**

Someone left this question blank!

5 ABC is a triangle inscribed in a circle. The tangent to the circle at A meets BC produced at D where BC = 10cm and AD = 12cm. What is the length of CD?



A. 6 cm

B. 7 cm

[✓ C.] 8 cm

D. 9 cm

$AD^2 = BD.CD$ (square of the tangent)

$\therefore 144 = (10 + CD)CD$

$\therefore CD^2 + 10CD - 144 = 0$

$\therefore (CD - 8)(CD + 18) = 0$

$\therefore CD = 8$

A	9
B	6
C	144
D	1

Someone left this question blank!

6 Which of the following is the range of the function $y = 2\sin^{-1} x + \frac{\pi}{2}$?

A. $-\pi \le y \le \pi$

B. $-\pi \le y \le \frac{3\pi}{2}$

C. $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

[✓ D.] $-\frac{\pi}{2} \le y \le \frac{3\pi}{2}$

$$-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}$$

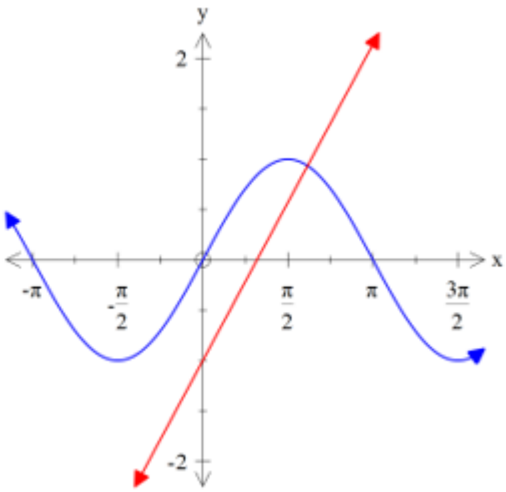
$$\therefore -\frac{\pi}{2} \times 2 + \frac{\pi}{2} \le 2\sin^{-1} x + \frac{\pi}{2} \le 2 \times \frac{\pi}{2} + \frac{\pi}{2}$$

$$\therefore -\frac{\pi}{2} \le 2\sin^{-1} x + \frac{\pi}{2} \le \frac{3\pi}{2}$$

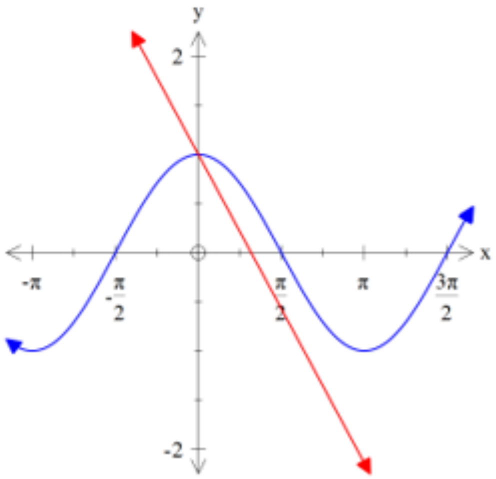
A	9
B	6
C	0
D	146

7 Which of the following graphs could be used to solve $\cos x + x = 0$?

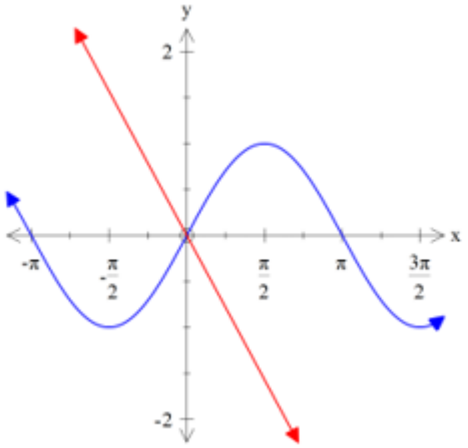
A.



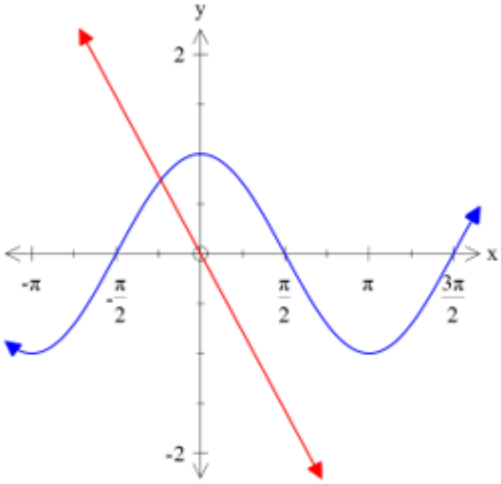
B.



C.



[✓ D.]



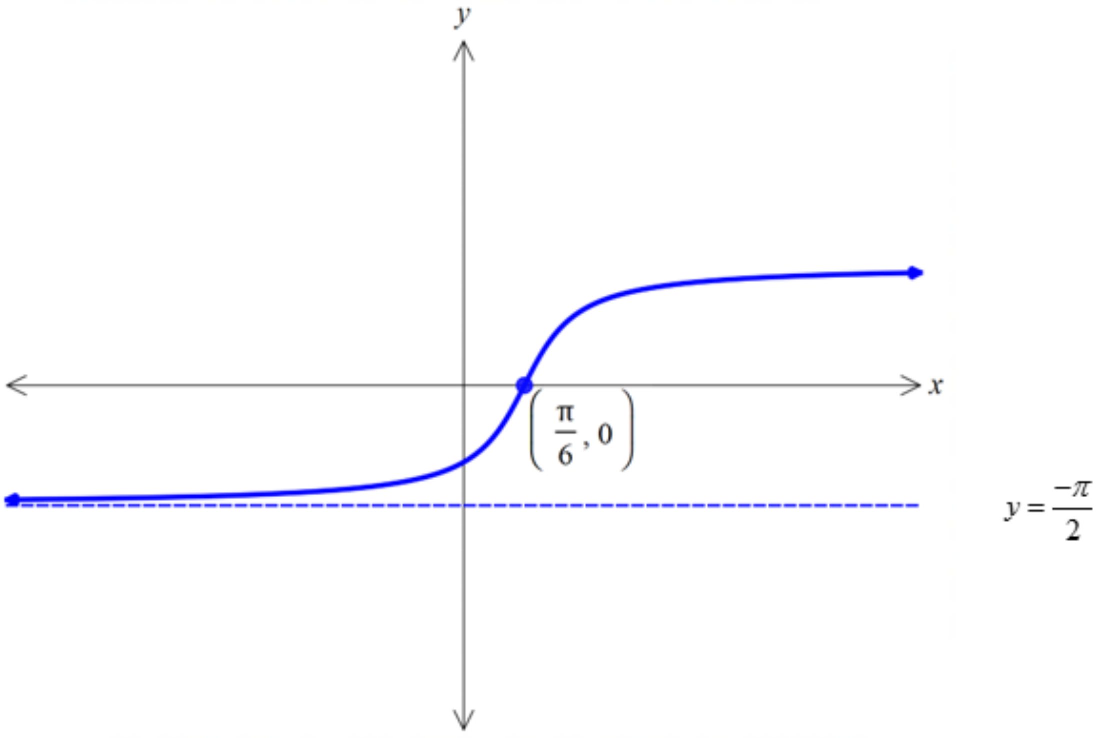
A has the graph of $\sin x$

B has the graph of $y = 1 - x$

C has the graph of $\sin x$

A	2
B	10
C	11
D	138

8 Let k be a positive constant and $-\pi \le \theta \le \pi$. If the diagram shows the graph of $y = \tan^{-1}(kx + \theta)$, then:



A. $k = -3$ and $\theta = \frac{\pi}{2}$

B. $k = \frac{1}{3}$ and $\theta = \frac{\pi}{2}$

[✓ C.] $k = 3$ and $\theta = -\frac{\pi}{2}$

D. $k = \frac{-1}{3}$ and $\theta = -\frac{\pi}{2}$

A	25
B	6
C	127
D	3

The graph has shifted $\frac{\pi}{6}$ units to the right.

Now $\tan^{-1}(kx + \theta) = \tan^{-1} k(x + \tfrac{\theta}{k})$ and so $\tfrac{\theta}{k}$ must be negative.

As $k > 0$ then $\theta < 0$.

9 Find $\sin\left(2\tan^{-1}\frac{1}{2}\right)$

A. $\frac{5}{4}$

B. $\frac{3}{5}$

[✓ C.] $\frac{4}{5}$

D. $\frac{5}{3}$

1st method: Use calculator

Slow method: Let $\alpha = \tan^{-1}\frac{1}{2} \Rightarrow \tan\alpha = \frac{1}{2}$

$$\therefore \sin\alpha = \tfrac{1}{\sqrt{5}}, \cos\alpha = \tfrac{2}{\sqrt{5}}$$

$$\begin{aligned}\therefore \sin(2\alpha) &= 2\sin\alpha\cos\alpha \\ &= 2\times\tfrac{1}{\sqrt{5}}\times\tfrac{2}{\sqrt{5}} \\ &= \tfrac{4}{5}\end{aligned}$$

Obviously A and D are clearly wrong as $-1 \le \sin\theta \le 1$.

A	1
B	0
C	160
D	0

10 Finding $\lim\limits_{x\to 0}\frac{\sin 4x}{x}$ gives

A. $\frac{1}{4}$

[✓ B.] 4

C. 2π

D. $\frac{\pi}{8}$

1st method: B by inspection.

Slow method:

$$\begin{aligned}\lim_{x\to 0}\frac{\sin 4x}{x} &= 4\times\lim_{4x\to 0}\frac{\sin 4x}{4x} \\ &= 4\times 1 \\ &= 4\end{aligned}$$

A	7
B	154
C	0
D	0

Section II

Question 11.

a) i $\frac{x}{1+x^2}$ $-u$ $-v$ 　 $u = x$ 　 $v = 1 + x^2$

$u' = 1$ 　 $v' = 2x$

$\frac{d}{dx} = \frac{(1)(1+x^2) - x(2x)}{(1+x^2)^2}$ ①

Generally well done.

$= \frac{1 + x^2 - 2x^2}{(1+x^2)^2}$

$= \frac{1-x^2}{(1+x^2)^2}$ ②

A few Ss are putting +C on their derivatives.

ii $\sin(\cos^2 x + e^x)$

$\frac{d}{dx} = \cos(\cos^2 x + e^x) \cdot \frac{d}{dx}(\cos^2 x + e^x)$ ①

$= \cos(\cos^2 x + e^x) \cdot (-\sin 2x + e^x)$

$= \cos(\cos^2 x + e^x)(e^x - \sin 2x)$ ②

missing brackets $-\frac{1}{2}$

b) $\int_0^1 \left(e^{-x} + \frac{1}{1+x} + \frac{1}{\sqrt{(1-x^2)}} \right) dx$

common errors: $\tan^{-1} x$ or $\ln x$ instead of $\ln(1+x)$

$= \left[-e^{-x} + \ln(1+x) + \sin^{-1} x \right]_0^1$ ①

$= \left[-e^{-1} + \ln 2 + \sin^{-1} 1 \right] - \left[-e^0 + \ln(1) + \sin^{-1} 0 \right]$

$= \left[-\frac{1}{e} + \ln 2 + \frac{\pi}{2} \right] - \left[-1 + 0 + 0 \right]$

$= 1 + \ln 2 + \frac{\pi}{2} - \frac{1}{e}$ ②

c) $\int_0^1 \frac{x^3}{1+x^8}\,dx$

$u = x^4$

$x = 1 \Rightarrow u = 1$

$x = 0 \Rightarrow u = 0$

$= \frac{1}{4}\int_0^1 \frac{4x^3}{1+x^8}\,dx$

$\frac{du}{dx} = 4x^3$

$du = 4x^3\,dx$

$= \frac{1}{4}\int_0^1 \frac{du}{1+u^2}$ ①

$= \frac{1}{4}\Big[\tan^{-1}u\Big]_0^1$ ②

Generally well done.

$= \frac{1}{4}\left[\frac{\pi}{4} - 0\right]$

$= \frac{\pi}{16}$ ③

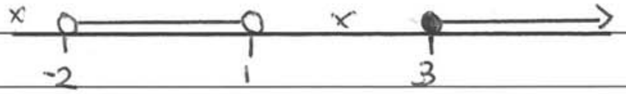
d) $\frac{x-3}{(x^2+x-2)} \geqslant 0$

$\frac{x-3}{(x+2)(x-1)} \geqslant 0$

$\therefore x \neq -2, 1$

$\frac{x-3}{(x+2)(x-1)} \times (x+2)^2(x-1)^2 \geqslant 0 \times \left[(x+2)^2(x-1)^2\right]$

$(x-3)(x+2)(x-1) \geqslant 0$



②

$-2 < x < 1, \quad x \geqslant 3$

-1 for $-2 \leq x \leq 1$

-1 for missing $x \geqslant 3$

e) ∠ of projection = 0
initial velocity = 80

horizontal	vertical
$\ddot{x} = 0$	$\ddot{y} = -10$
$\dot{x} = 80$	$\dot{y} = -10t$
$x = 80t$	$y_1 = -5t^2 + 100$
	$y_2 = -5t^2 + 200$ ①

when $y = 0$

$-5t^2 + 100 = 0$
$5t^2 = 100$
$t^2 = 20$
$t = \sqrt{20}$ $t \neq$ neg.
$= 2\sqrt{5}$

$-5t^2 + 200 = 0$
$5t^2 = 200$
$t^2 = 40$ ②
$t = \sqrt{40}$
$= 2\sqrt{10}$

$x_1 = 80t$
$= 80 \times 2\sqrt{5}$
$= 160\sqrt{5}$

$x_2 = 80t$
$= 80 \times 2\sqrt{10}$ ③
$= 160\sqrt{10}$

distance apart $= 160\sqrt{10} - 160\sqrt{5}$
$= 160(\sqrt{10} - \sqrt{5})$
$= 160\sqrt{5}(\sqrt{2} - 1)$ ④
$\doteqdot 148.19$

Some Ss didn't realise $\theta = 0$
Many rounded decimals - better to leave as exact value.
fairly well done.

Q12 XI

(a) $\int_0^{\frac{\pi}{4}} f(x)\,dx = 5$ and $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} f(x)\,dx = 2 \Rightarrow \int_0^{\frac{\pi}{2}} f(x)\,dx = 7$

Find a if $\int_{\frac{\pi}{2}}^{0} \left(f(x) + a\sin 2x\right) dx = 10$

$$\Rightarrow \int_{\frac{\pi}{2}}^{0} f(x)\,dx + a\int \sin 2x\,dx = 10$$

$$\Rightarrow -\int_0^{\frac{\pi}{2}} f(x)\,dx + a\int_{\frac{\pi}{2}}^{0} \sin 2x\,dx = 10$$

$$\Rightarrow -7 + a\left[\frac{-\cos 2x}{2}\right]_{\frac{\pi}{2}}^{0} = 10$$

$$-a\left(\frac{1}{2} - -\frac{1}{2}\right) = 17$$

$$\Rightarrow -a = 17 \qquad \underline{\underline{a = -17}}$$

②

Many careless errors with this question, including forgetting the negative sign when integrating and not noting the order of the bounds in the given information.

12 (b)(i)

For $(2+x)^n$, $T_{n+1} = {}^nC_r 2^{n-r} x^r$

Let $r=0 \Rightarrow T_{n+1} = {}^nC_0 2^n = 2^n$

For $(2-ax)^{2n+1}$, $T_{n+1} = {}^{2n+1}C_r 2^{2n+1-r} (ax)^r (-1)^r$

Let $r=0 \Rightarrow T_{n+1} = {}^{2n+1}C_0 2^{2n+1} = 2^{2n+1}$ ✓

Then $\frac{2^n}{2^{2n+1}} = \frac{1}{8}$

$2^{-n-1} = 2^{-3}$ ✓ ②

This question was done well.

$\Rightarrow \underline{n=2}$

(ii) For $(1+x)(2-ax)^5$ ✓

Term in $x^2 = (1 \times {}^5C_2 2^3 a^2 x^2) + (x \times {}^5C_1 2^4 a (-1) x)$

$\Rightarrow 80a^2 - 80a = 160$ ✓

$a^2 - a - 2 = 0$

$(a-2)(a+1) = 0$ ③

$\Rightarrow \underline{a = 2 \text{ or } -1}$ ✓

If students could find the term in x^2 then they generally did this question correctly.

12 (c)(i) Show $\frac{\sin^3\theta+\cos^3\theta}{\sin\theta+\cos\theta} = \frac{2-\sin 2\theta}{2}$

$$LHS = \frac{(\cancel{\sin\theta+\cos\theta})(\sin^2\theta-\sin\theta\cos\theta+\cos^2\theta)}{\cancel{\sin\theta+\cos\theta}}$$

$$= 1-\sin\theta\cos\theta$$

$$= \frac{2-2\sin\theta\cos\theta}{2}$$

$$= \frac{2-\sin 2\theta}{2}.$$

✓ ②

About half of students did not recognise that they could factorise the sum of 2 cubes. In that case they had to do a lot more work for the 2 marks and often got into difficulty.

(ii) $\frac{2-\sin 2\theta}{2} = \frac{7}{10}$

$$20-10\sin 2\theta = 14$$

$$-10\sin 2\theta = -6$$

$$\sin 2\theta = 3/5$$

✓ ①

This question was done very well.

(iii) $\tan 3\theta = \tan\left(\frac{3}{2}\times 2\theta\right)$

$$= \tan\left(\frac{3}{2}\sin^{-1}(3/5)\right)$$

$$= \frac{13}{9}$$

✓ ②

Lots of students took this direct approach but many used the compound angle formula for tan and usually got into trouble with it.

(d) $3\sin\theta - 4\cos\theta = 4$

$$3\times\frac{2t}{1+t^2} - \frac{4(1-t^2)}{1+t^2} = 4$$

$$6t-4+4t^2 = 4+4t^2$$

$$6t = 8$$

$$t = \frac{4}{3}$$

$$\Rightarrow \tan\frac{\theta}{2} = \frac{4}{3}$$

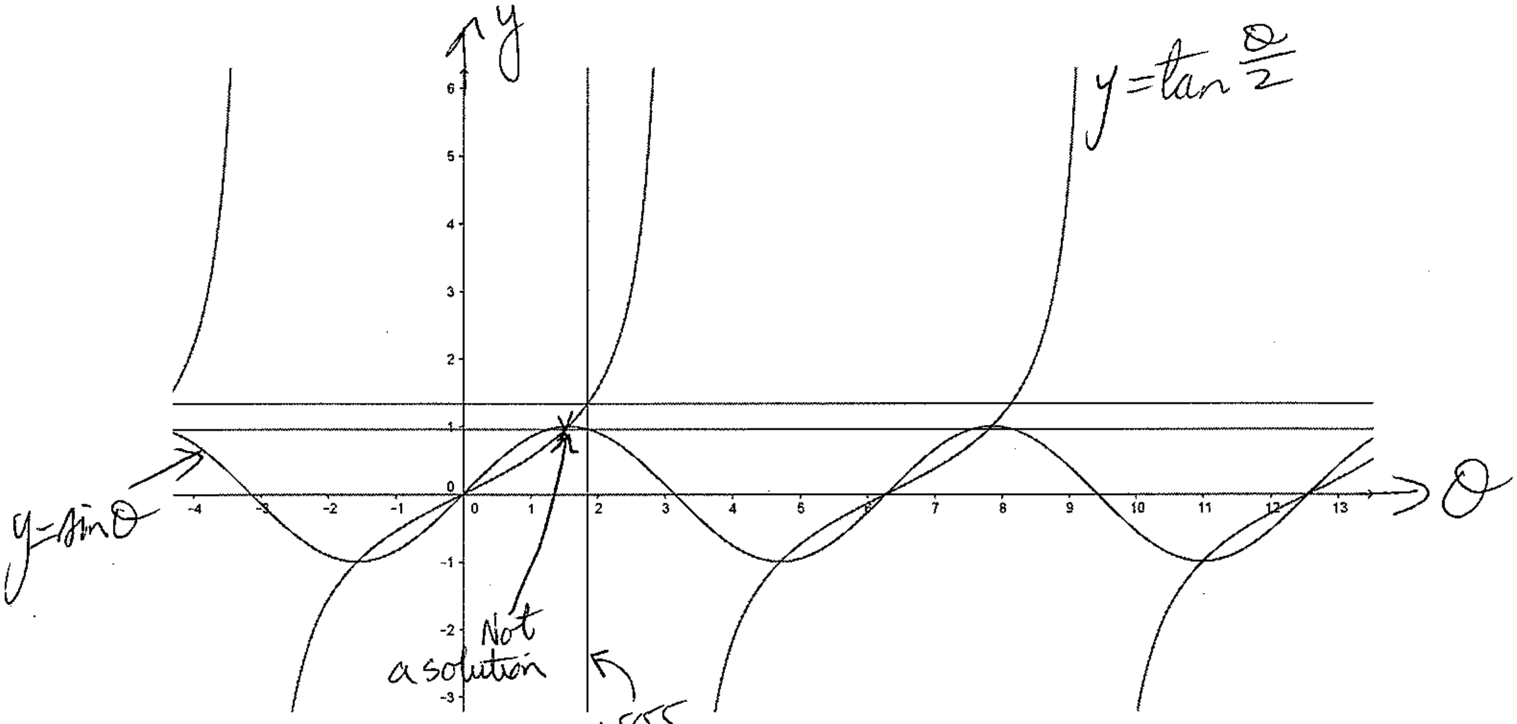
$$\frac{\theta}{2} = 0.92729 \text{ or } (\pi+.92729)$$

too big

$$\Rightarrow \theta = 1.855$$ ✓ Also check $\theta = \pi$

Yes $\theta = \pi$ is a soln too. ✓ ③

Some students substituted back into the sine identity in terms of t. This threw up extra solutions which were not solutions of the original tan equation. See graph below..



Not a solution

y = 1.855 radians is the solution.

13) a) consider $\left(\frac{2}{x} - x\right)^6$

$$T_{k+1} = {}^6C_k \left(\frac{2}{x}\right)^{6-k} (-x)^k$$

$$= {}^6C_k (2)^{6-k} (-1)^k (x^{-1})^{6-k} (x)^k$$

$$= {}^6C_k (2)^{6-k} (-1)^k x^{k-6} . x^k$$

$$= {}^6C_k (2)^{6-k} (-1)^k x^{2k-6}$$

let $2k-6=-2$

$2k=4$

$k=2$

co.eff of x^{-2} is ${}^6C_2 . 2^{6-2} . (-1)^2 = 240$

consider $\left(1+\frac{2}{x}\right)^n$

co.eff of x^{-2} is ${}^nC_2 . 2^2 = 4. {}^nC_2$

$\therefore\ 240 - 4. {}^nC_2 = 128$

$4. {}^nC_2 = 112$

${}^nC_2 = 28$

$$\frac{n!}{(n-2)!\,2!} = 28$$

$$\frac{n(n-1)}{2} = 28$$

$n(n-1) = 56$

$n^2 - n - 56 = 0$

$(n-8)(n+7) = 0$

$n = -7, 8$

since n is a positive integer

$\underline{n=8}$

Comment: Marks were easily gained by knowing the binomial expansion.

b) i) $4ay = x^2$

$y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{2x}{4a}$

$= \frac{x}{2a}$

when $x = 2a\cancel{t}$

$m_T = \frac{(2a\cancel{t})}{2a}$

$= t$

$y - y_1 = m(x - x_1)$

$y - at^2 = \cancel{2a}t(x - 2at)$

$y - at^2 = tx - 2at^2$

$y = tx - at^2$

ii) let $y = 0$

$0 = tx - at^2$

$tx = at^2$

$x = at$

$\therefore A(at, 0)$

let $x = 0$

$y = -at^2$

$\therefore B(0, -at^2)$

iii) $A(at, 0)$ $B(0, -at^2)$

$1 : m$

$\frac{m(at) + 1.(0)}{1+m} = 2at$

$$\frac{m\cancel{at}}{1+m} = 2\cancel{at}$$

$$m = 2 + 2m$$

$$m = -2$$

$\therefore$ P divides AB in the ratio $1:-2$

ie externally in the ratio $1:2$

NOTE: Many students started with the ratio $m:n$ which led to the result $n=-2m$. The most common mistake was to then say the ratio is $2:1$ (external)

Let's check! $m:n$

$$\cancel{m} : -2\cancel{m}$$

$$1 : -2$$

iv) $S(0,a)$ $\quad P(2at, at^2)$

$$m_{PS} = \frac{at^2 - a}{2at - 0}$$

$$= \frac{\cancel{a}(t^2-1)}{2\cancel{a}t}$$

$$= \frac{t^2-1}{2t}$$

v) let $\angle SPB = \theta$

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

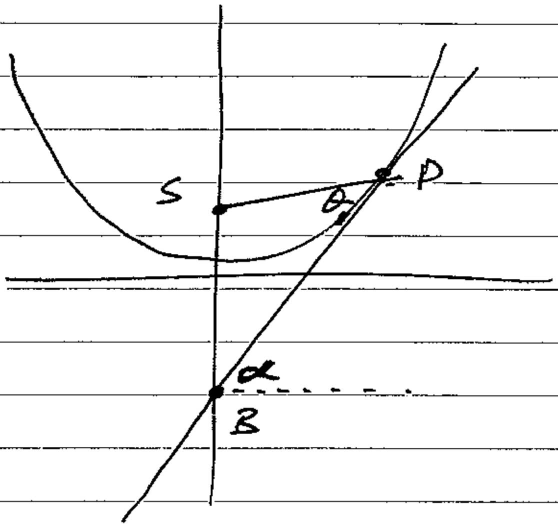
$$\tan\theta = \left|\frac{t - \frac{t^2-1}{2t}}{1 + t.\frac{t^2-1}{2t}}\right|$$

$$\tan\theta = \left|\frac{2t^2 - t^2 + 1}{2t + t^3 - t}\right|$$

$$\tan\theta = \left|\frac{\cancel{t^2+1}}{t\cancel{(t^2+1)}}\right|$$

$$= \left|\frac{1}{t}\right|$$

$$\therefore \theta = \tan^{-1}\left|\frac{1}{t}\right|$$



$\tan\alpha = t \quad (m_{PB})$

$\angle SBP = \frac{\pi}{2} - \alpha$

$$\tan\left(\frac{\pi}{2} - \alpha\right) = \cot\alpha$$

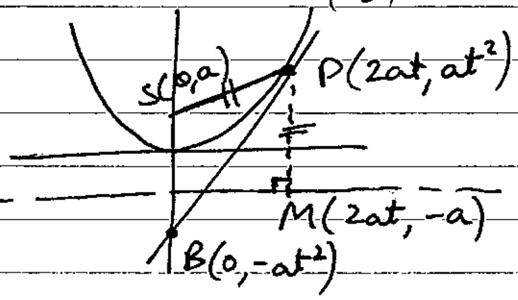
$$= \frac{1}{\tan\alpha}$$

$$= \frac{1}{t}$$

$$\frac{\pi}{2} - \alpha = \tan^{-1}\left|\frac{1}{t}\right|$$

$\therefore \angle SPB = \angle PBS$

OR



$PS = PM$ (definition of a parabola)

$= at^2 - (-a)$

$= a(t^2+1)$

$BS = a - (-at^2)$

$= a(t^2+1)$

$PS = BS$

$\therefore \angle SPB = \angle PBS$ (opposite equal sides, $\triangle PBS$)

ie ℓ makes equal angles with y-axis and PS.

Comment: students should be familiar with this result and know how to prove it.

This wasn't a hence, show that question so there is no reason why students couldn't use the second method shown.

Overall this should have been 8 relatively easy marks for a pretty routine question.

c) P(3 fail) $= {}^{10}C_3\left(\frac{9}{10}\right)^7\left(\frac{1}{10}\right)^3$

$$= \frac{10!}{3!(10-3)!}\;\frac{9^7}{10^{10}}$$

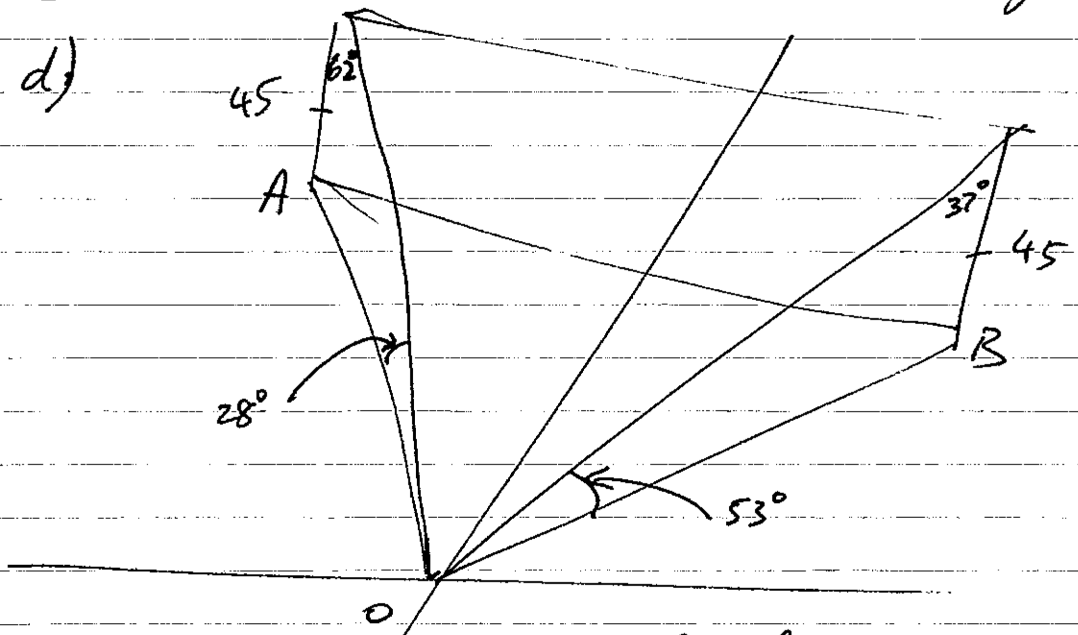
$$= \frac{10.\cancel{9}^{3}\cancel{8}^{4}}{\cancel{3}.\cancel{2}}\cdot\frac{(3^2)^7}{(2.5)^{10}}$$

$$= \frac{3^{15}}{2^7.5^9}$$

$$= \frac{14348907}{250000000}$$

Comment: This was done well in general.

d)



$\tan 62° = \frac{OA}{45}$

$OA = 45\tan 62$

$\tan 37 = \frac{OB}{45}$

$OB = 45\tan 37$

$\angle AOB = 25 + (360 - 335)$
$= 50^\circ$

$AB^2 = OA^2 + OB^2 - 2.OA.OB.\cos 50^\circ$

$= (45\tan 62)^2 + (45\tan 37)^2 - 2(45\tan 62)(45\tan 37).\cos 50$

$= 4623.117\ldots$

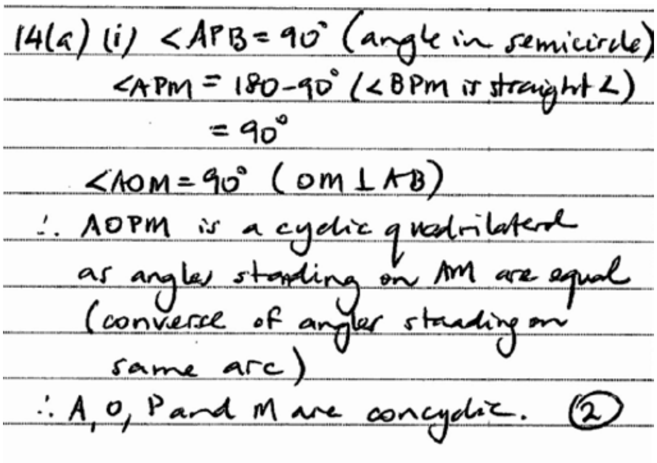
$\underline{AB \approx 68\text{ m}}$

Comment: Another routine problem. Although students weren't led through the question, they should be familiar with the technique of bringing the height down to a horizontal triangle.

Note: By using complementary angles the working is simplified somewhat.

Ext 1 Y12 THSC 2018 Q14 solutions

Mean (out of 16): 9.02



14(a) (i) $\angle APB = 90^\circ$ (angle in semicircle)

$\angle APM = 180 - 90^\circ$ ($\angle BPM$ is straight $\angle$)

$= 90^\circ$

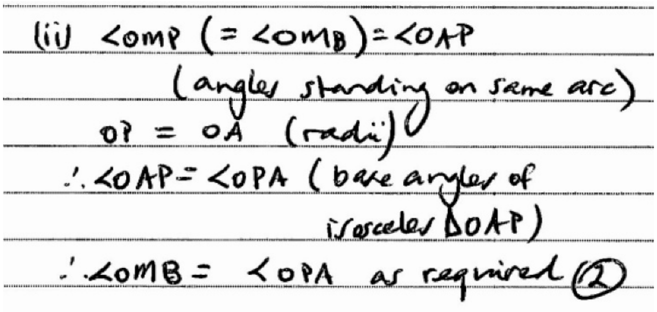
$\angle AOM = 90^\circ$ ($OM \perp AB$)

$\therefore$ AOPM is a cyclic quadrilateral as angles standing on AM are equal (converse of angles standing on same arc)

$\therefore$ A, O, P and M are concyclic. ②

Many students referred to reasons such as "angles in the same segment" even though it had not yet been determined that the points were concyclic. Students should refer to the converse of such results. On this occasion, students were not penalised.

0	0.5	1	1.5	2	Mean
24	18	10	12	97	1.43



(ii) $\angle OMP$ ($= \angle OMB$) $= \angle OAP$ (angles standing on same arc)

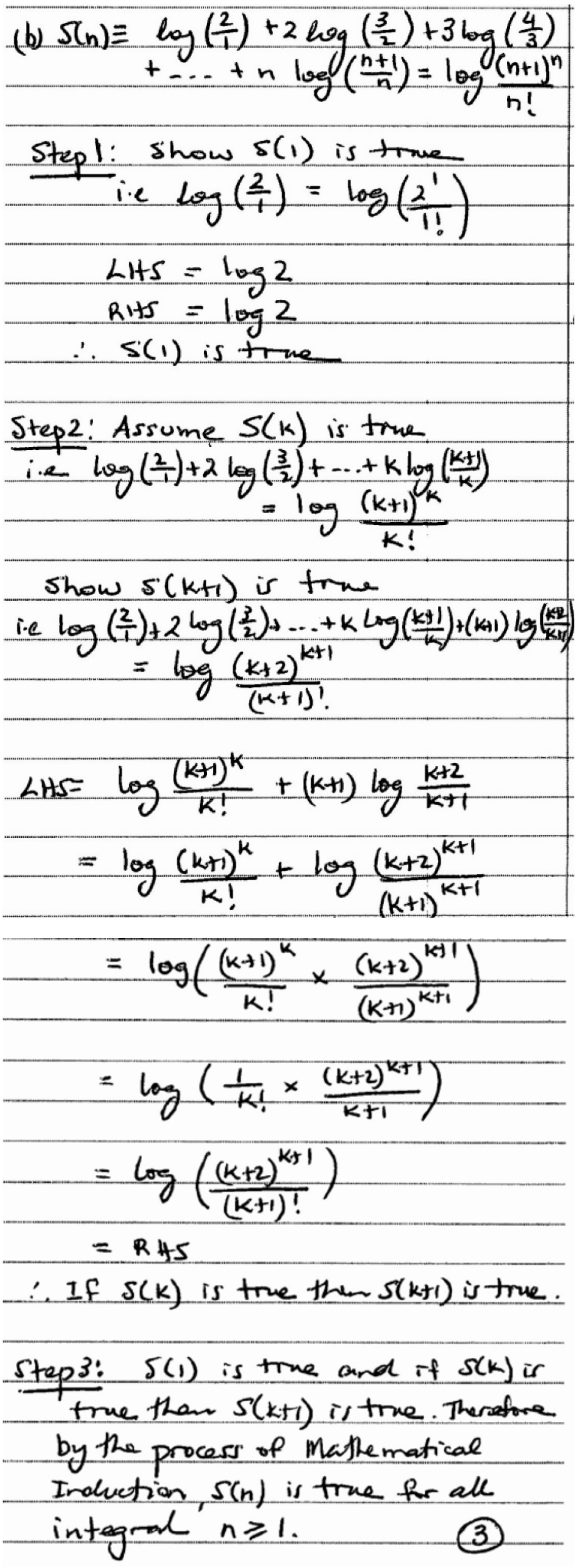
$OP = OA$ (radii)

$\therefore \angle OAP = \angle OPA$ (base angles of isosceles $\triangle OAP$)

$\therefore \angle OMB = \angle OPA$ as required ②

One student used the elegant proof that the required angles stood on the chords OA and OP (which are radii of the original circle and therefore equal) and hence the angles at the circumference of the newly identified circle must be equal.

0	0.5	1	1.5	2	Mean
34	7	6	4	110	1.46



(b) $S(n) \equiv \log\left(\frac{2}{1}\right) + 2\log\left(\frac{3}{2}\right) + 3\log\left(\frac{4}{3}\right) + \ldots + n\log\left(\frac{n+1}{n}\right) = \log\frac{(n+1)^n}{n!}$

Step 1: Show $S(1)$ is true

i.e $\log\left(\frac{2}{1}\right) = \log\left(\frac{2^1}{1!}\right)$

LHS $= \log 2$

RHS $= \log 2$

$\therefore S(1)$ is true

Step 2: Assume $S(k)$ is true

i.e $\log\left(\frac{2}{1}\right) + 2\log\left(\frac{3}{2}\right) + \ldots + k\log\left(\frac{k+1}{k}\right) = \log\frac{(k+1)^k}{k!}$

Show $S(k+1)$ is true

i.e $\log\left(\frac{2}{1}\right) + 2\log\left(\frac{3}{2}\right) + \ldots + k\log\left(\frac{k+1}{k}\right) + (k+1)\log\left(\frac{k+2}{k+1}\right) = \log\frac{(k+2)^{k+1}}{(k+1)!}$

$$\text{LHS} = \log\frac{(k+1)^k}{k!} + (k+1)\log\frac{k+2}{k+1}$$

$$= \log\frac{(k+1)^k}{k!} + \log\frac{(k+2)^{k+1}}{(k+1)^{k+1}}$$

$$= \log\left(\frac{(k+1)^k}{k!} \times \frac{(k+2)^{k+1}}{(k+1)^{k+1}}\right)$$

$$= \log\left(\frac{1}{k!} \times \frac{(k+2)^{k+1}}{k+1}\right)$$

$$= \log\left(\frac{(k+2)^{k+1}}{(k+1)!}\right)$$

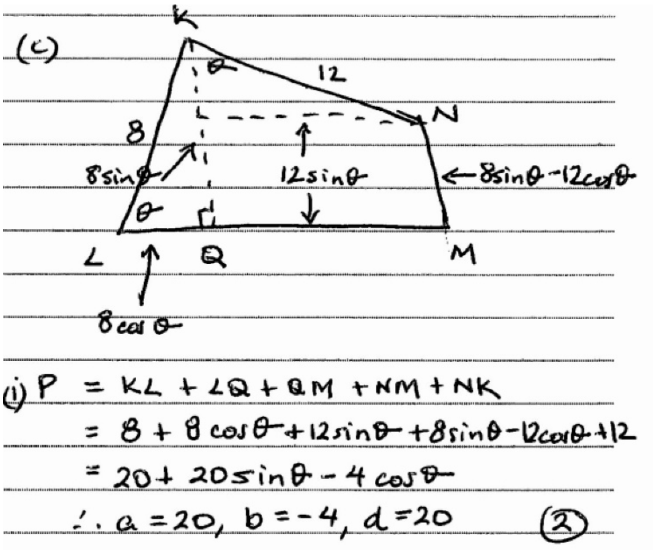
$= \text{RHS}$

$\therefore$ If $S(k)$ is true then $S(k+1)$ is true.

Step 3: $S(1)$ is true and if $S(k)$ is true then $S(k+1)$ is true. Therefore by the process of Mathematical Induction, $S(n)$ is true for all integral $n \geq 1$. ③

This was poorly done. Some students could not identify the general term of the series ($k\log\frac{k+1}{k}$), some thought that the target expression was $\frac{\log(k+1)^k}{k!}$ rather than $\log\frac{(k+1)^k}{k!}$. Students who could identify the general term and who used the correct target generally were successful. Some arguments used were sloppy. Some wrote things like “Assume *n* = *k*” and then “Prove *n* = *k* + 1”. If you assume that *n* = *k* then it follows that *n* + 1 = *k* + 1.

0	0.5	1	1.5	2	2.5	3	Mean
11	56	16	0	0	4	74	1.71



(c)

(i) $P = KL + LQ + QM + NM + NK$

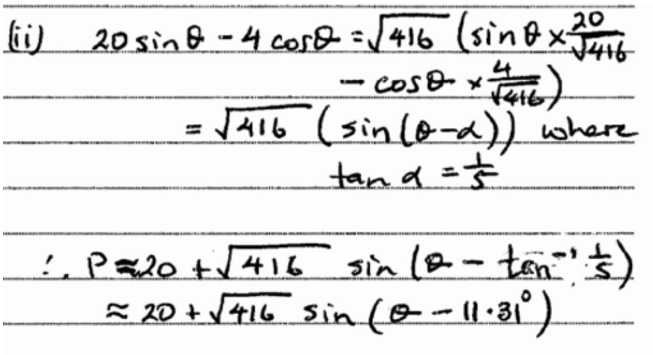
$= 8 + 8\cos\theta + 12\sin\theta + 8\sin\theta - 12\cos\theta + 12$

$= 20 + 20\sin\theta - 4\cos\theta$

$\therefore a = 20,\ b = -4,\ d = 20$ ②

This was found to be quite difficult. The supplied shape allowed expressions for each of the components of the perimeter to be found in terms of $\sin\theta$ and $\cos\theta$.

0	0.5	1	1.5	2	Mean
57	18	5	3	78	1.08



(ii) $20\sin\theta - 4\cos\theta = \sqrt{416}\left(\sin\theta \times \frac{20}{\sqrt{416}} - \cos\theta \times \frac{4}{\sqrt{416}}\right)$

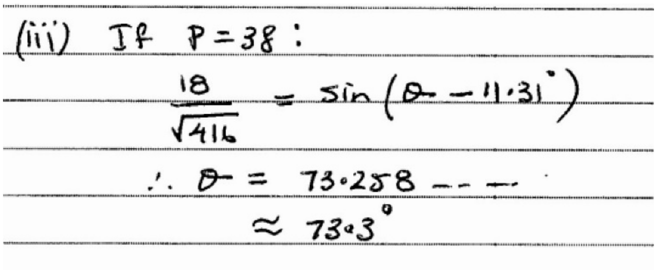
$= \sqrt{416}\,(\sin(\theta - \alpha))$ where $\tan\alpha = \frac{1}{5}$

$\therefore P \approx 20 + \sqrt{416}\ \sin\left(\theta - \tan^{-1}\frac{1}{5}\right)$

$\approx 20 + \sqrt{416}\ \sin(\theta - 11.31^\circ)$

Students who came out with some expression for the perimeter in part (i) were normally able to generate appropriate expressions for P.

0	0.5	1	1.5	2	Mean
61	11	12	15	62	1.02



(iii) If $P = 38$:

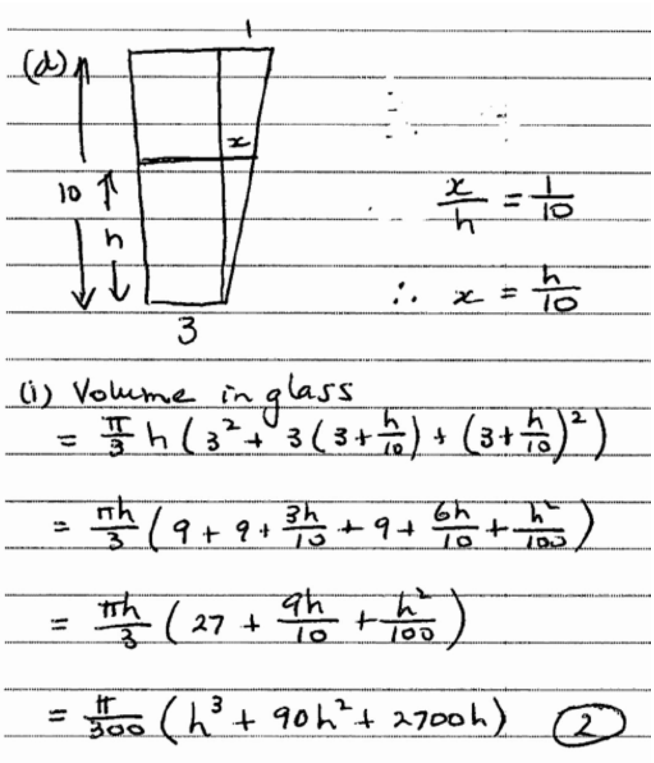
$\frac{18}{\sqrt{416}} = \sin(\theta - 11.31^\circ)$

$\therefore \theta = 73.258 \ldots$

$\approx 73.3^\circ$

Students who came up with an expression in part (ii) could normally calculate a value for θ. (However, sometimes the expression determined in part (ii) led to a value for $\sin\theta$ which was greater than 1. This did not seem to trouble anyone.)

0	0.5	1	Mean
81	18	62	0.44



(d)

$\frac{x}{h} = \frac{1}{10}$

$\therefore x = \frac{h}{10}$

(i) Volume in glass

$= \frac{\pi}{3} h \left(3^2 + 3\left(3 + \frac{h}{10}\right) + \left(3 + \frac{h}{10}\right)^2\right)$

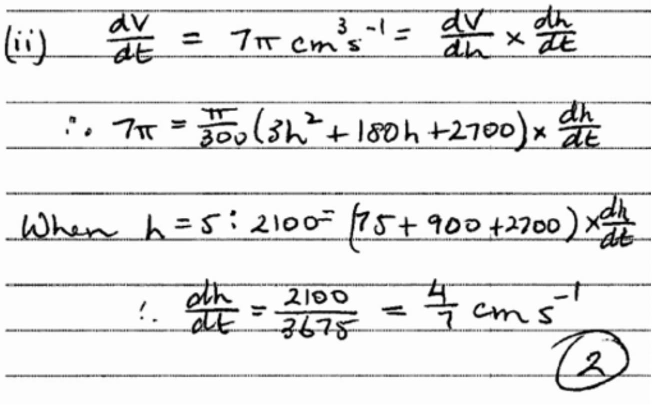
$= \frac{\pi h}{3}\left(9 + 9 + \frac{3h}{10} + 9 + \frac{6h}{10} + \frac{h^2}{100}\right)$

$= \frac{\pi h}{3}\left(27 + \frac{9h}{10} + \frac{h^2}{100}\right)$

$= \frac{\pi}{300}\left(h^3 + 90h^2 + 2700h\right)$ ②

Students who found an appropriate expression for the radius of the surface of the liquid generally succeeded. A few students decided that they would ignore the supplied formula and find the volume of required frustum from scratch, deducing that the heights of the associated cones were 30 and $30 + h$. They normally arrived at the correct equation.

0	0.5	1	1.5	2	Mean
97	17	5	1	41	0.60



(ii) $\frac{dV}{dt} = 7\pi \text{ cm}^3\text{s}^{-1} = \frac{dV}{dh} \times \frac{dh}{dt}$

$\therefore 7\pi = \frac{\pi}{300}(3h^2 + 180h + 2700) \times \frac{dh}{dt}$

When $h = 5$: $2100 = (75 + 900 + 2700) \times \frac{dh}{dt}$

$\therefore \frac{dh}{dt} = \frac{2100}{3675} = \frac{4}{7} \text{ cm s}^{-1}$

②

This was normally done well by students who attempted it using the supplied formula for the volume in part (i).

0	0.5	1	1.5	2	Mean
49	5	7	11	89	1.27